

# Geometrical string and dual spin systems \*

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## Abstract

We are able to perform the duality transformation of the spin system which was found before as a lattice realization of the string with linear action. In four and higher dimensions this spin system can be described in terms of a two-plaquette gauge Hamiltonian. The duality transformation is constructed in geometrical and algebraic language. The dual Hamiltonian represents a new type of spin system with local gauge invariance. At each vertex  $\xi$  there are  $d(d-1)/2$  Ising spins  $\Lambda_{\mu,\nu} = \Lambda_{\nu,\mu}$ ,  $\mu \neq \nu = 1, \dots, d$  and one Ising spin  $\Gamma$  on every link  $(\xi, \xi + e_\mu)$ . For the frozen spin  $\Gamma \equiv 1$  the dual Hamiltonian factorizes into  $d(d-1)/2$  two-dimensional Ising ferromagnets and into antiferromagnets in the case  $\Gamma \equiv -1$ . For fluctuating  $\Gamma$  it is a sort of spin glass system with local gauge invariance. The generalization to  $p$ -branes is given.

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\*Dedicated to Professor Herbert Wagner at the occasion of his 60th birthday

# 1 Introduction

## 1.1 String with linear action

The strong coupling expansion of the lattice gauge theories and topological classification of the QCD diagrams, together with the last progress in solving two-dimensional QCD, put forward the ultimate conjunction that QCD is exactly equivalent to a string theory [1]. One can also expect a new progress in understanding of the QCD string in four dimensions [2, 3]. In this approach one should derive the relevant string theory directly from properly regularized QCD path integrals.

In the complementary approach [4, 5, 6, 7] one can try to derive the relevant string Lagrangian from alternative principles such as [8] :

$\alpha$ ) natural coincidence of the string transition amplitude with the usual Feynman path amplitude for long space-time strips and

$\beta$ ) the continuity principle for the string amplitudes.

The partition function for the regularized string, which can be derived from these principles has the form

$$Z(\beta) = \sum_{\{M\}} \exp\{-\beta A(M)\}, \quad (1)$$

where the summation is extended over all triangulated surfaces  $\{M\}$  with the linear action  $A(M)$

$$A(M) = \sum_{\langle i,j \rangle} \lambda_{i,j} \cdot |\pi - \alpha_{i,j}| \quad (2)$$

and  $\alpha_{i,j}$  is the angle between two neighbor faces of  $M$  having a common edge  $\langle i, j \rangle$  of the length  $\lambda_{i,j}$ . The regularized string (1), (2) is well defined in any dimensions and for arbitrary topology of the surface  $M$ .

## 1.2 Surfaces on the lattice

In a recent paper [9] two of the authors have introduced a spin statistical system on the lattice, the *low* temperature expansion of which generates random surfaces with linear action  $A(M)$  (2). In this lattice implementation of the linear string the corresponding Hamiltonian depends on the way how we ascribe the weights on the self-intersection edges.

Indeed as it is well known, the surfaces of interface naturally appearing in the low temperature expansion of the partition function have self-intersections, and it is an important problem to find out a way to treat them properly. In general one can define corresponding weights at the self-intersection edges in different ways without spoiling the linear character of the theory (2).

According to  $\alpha$ ) and  $\beta$ ) one should count the angles between plaquettes approaching the self-intersection edges and then multiply the result by  $\lambda_{i,j}$  [8]

$$\lambda_{i,j} \cdot (|\pi - \alpha_{i,j}| + |\pi - \beta_{i,j}| + \dots). \quad (3)$$

In [9] the authors suggested to ascribe the weights equal to the length  $\lambda_{i,j}$  multiplied by the *total* number of the right angles plaquettes approaching the edge  $\langle i, j \rangle$ .

One can also consider these weights with arbitrary self-intersection coupling constant  $k$  and observe that in the limit, when the self-intersection coupling constant  $k$  tends to infinity, we end up with self-avoiding surfaces [10]. It also became clear, that in the case of a three-dimensional lattice the system drastically simplifies in the opposite limit, when the self-intersection coupling constant  $k$  tends to zero. This simplification is strongly connected with an additional symmetry which appears in the system in this limit and allows therefore to construct the dual Hamiltonian.

In the following we will first reconsider the three-dimensional model and its dual, in section 2 surfaces with linear action in four dimensions are considered, and in section 3 a formulation for general dimensions will be given.

### 1.3 Duality in three dimensions

In the limit, when self-intersection coupling constant  $k$  tends to zero, the additional symmetry and the positivity of the local weights allows to construct the dual Hamiltonian in three dimensions which generates random surfaces with linear action  $A(M)$  now in its *high* temperature expansion.

In terms of Ising spin variables  $\sigma_{\vec{r}}$  the Hamiltonian with self-intersection coupling constant  $k$  equal to zero has the form

$$H_{gonihedric}^{3d} = - \sum_{\vec{r}, \vec{\alpha}, \vec{\beta}} \sigma_{\vec{r}} \sigma_{\vec{r} + \vec{\alpha}} \sigma_{\vec{r} + \vec{\alpha} + \vec{\beta}} \sigma_{\vec{r} + \vec{\beta}} \quad (4)$$

where  $\vec{r}$  is a three-dimensional vector on the lattice and  $\vec{\alpha}, \vec{\beta}$  are unit vectors parallel to the axes. We should stress that the Ising spins in (4) are on the vertices of a simple cubic lattice and are not on the links. The details of the construction of the Hamiltonian (4) from the first principles can be found in [9, 10, 12]. In addition to the usual symmetry group  $Z_2$  the system (4) has a new symmetry: one can independently flip the spins on any combination of planes (spin layers) of the three-dimensional lattice.

The dual Hamiltonian reads [12]

$$H_{dual}^{3d} = - \sum_{\xi} \Lambda^{\chi}(\xi) \Lambda^{\chi}(\xi + \chi) + \Lambda^{\eta}(\xi) \Lambda^{\eta}(\xi + \eta) + \Lambda^{\varsigma}(\xi) \Lambda^{\varsigma}(\xi + \varsigma), \quad (5)$$

where  $\chi, \eta$ , and  $\varsigma$  are unit vectors in the corresponding directions of the dual lattice and  $\Lambda$ 's are one-dimensional irreducible representations of the forth order Abelian group  $G_{\xi} = \{e, g_{\chi}, g_{\eta}, g_{\varsigma}\}$

$$\Lambda^{\chi} = \{1, 1, -1, -1\}, \quad \Lambda^{\eta} = \{1, -1, 1, -1\}, \quad \Lambda^{\varsigma} = \{1, -1, -1, 1\}. \quad (6)$$

Spin  $G_{\xi}$  should be ascribed to every vertex  $\xi$  of the dual lattice.  $\Lambda^{\chi}$  and  $\Lambda^{\eta}$  may also be considered as Ising spins, then  $\Lambda^{\varsigma} = \Lambda^{\chi} \Lambda^{\eta}$  and (6) describes a modified Ashkin-Teller model [22]. The partition function of the dual systems (5) can be written in the form

$$Z(\beta^*) = \sum_{\{G_{\xi}\}} \exp\{-\beta^* H_{dual}^{3d}\}. \quad (7)$$

Both Hamiltonian (4) and (5) generate random surfaces with linear-gonihedric action  $A(M)$  in *low* and *high* temperature expansion correspondingly.

One can compare this system with the 3d Ising ferromagnet

$$H_{Ising}^{3d} = - \sum_{link} \sigma \sigma \quad (8)$$

and with it's dual Hamiltonian [13]

$$H_{dual}^{3d} = - \sum_{plaquettes} \sigma \sigma \sigma \sigma \quad (9)$$

which generates random surfaces with area action.

## 1.4 String interpretation

There is an intensive study of the properties of cluster boundaries of the 3d Ising ferromagnet in order to get the meaning of random surfaces on the lattice governed by the area law [15, 16, 14]. It is also important to understand the phase structure of the linear systems (4) and (5) and to study the properties of the ensemble of random surfaces generated by this system. For that one can use the curvature representation of the linear action  $A(M)$  [11] and then find an equivalent representation of the partition function (1) in terms of propagation of the polygon-loops or string  $P$  in a given direction. The transition amplitude is equal to [12]

$$\exp\{-k(P) - l(P)\} \quad (10)$$

where  $k(P)$  is the total curvature of  $P$  and  $l(P)$  is the length. The interaction term is proportional to the overlapping length of the strings

$$A_{int} = l(P_1 \cap P_2). \quad (11)$$

In the first approximation ignoring the interaction term one can solve the system and see that it describes the propagation of almost free 2d Ising fermions [12]. This is a strong indication that the system has second order phase transition similar to 2d Ising ferromagnet.

Again comparing our system with 3d Ising ferromagnet one can see that the propagation of the string came up with the amplitude

$$\exp\{-l(P) - s(P)\} \quad (12)$$

where  $l(P)$  is the length and  $s(P)$  is the area. The interaction, which is now proportional to the overlapping area

$$A_{int} = s(P_1 \cap P_2) \quad (13)$$

is much stronger and probably cannot be represented as a propagation of the free string [15, 16]. There is an indication [17] that  $SU(N)$  lattice gauge theories with one plaquette action have similar representation, the interaction is also proportional to the overlapping area.

## 1.5 Duality in higher dimensions

Our aim is to construct dual representations of the linear system (1), (2) in four and higher dimensions. There is a new complication compared with the three-dimensional case where we have set the self-intersection coupling constant  $k$  equal to zero. Indeed if in four dimensions we set the self-intersection coupling constant  $k$  equal to zero then the Hamiltonian has one and three plaquette terms [10] and if we take  $k = 1$  [9] then it has one and two plaquette terms. It is almost impossible to work with these complicated Hamiltonians.

It is crucial at this point to use the lattice version of the continuity principle  $\beta$ ) in the following form : the self-intersection weight should be defined so that after pairwise connection of plaquettes at the intersection edge the weight remain the same. Indeed on a three-dimensional lattice the self-intersection of four plaquettes *uniquely* decomposes into two pairs of flat plaquettes in accordance with  $\beta$ ).

To satisfy the continuity principle in four- and higher dimensional lattices we will take weights in the intersections, so that plaquettes continuing straight across an edge do not contribute to the energy and only the number of plaquettes which do not continue in a flat two-dimensional plane do count. In this case we again have an additional symmetry and the positivity of the local weights which allows to construct the dual Hamiltonian. We will consider such a system in the next section.

## 2 Surfaces with linear action in four dimensions

On a four-dimensional simple hypercubic lattice one can have intersections of two, four and six plaquettes at an edge [9]. In accordance with  $\beta$ ) the intersection of six plaquettes contributes zero energy and can be uniquely decomposed into three flat pairs of plaquettes. The intersection of four plaquettes yields zero energy in the cases where the plaquettes lie in two flat planes and with the energy equal to  $\pi/2$  (see (2)) if a pair of plaquettes is left which does not lie in a plane. Therefore a four-plaquette intersection also uniquely decomposes into two flat planes in the first case and into one flat plane and one "corner" in the second case.

For this choice of the self-intersection weights the Hamiltonian has the form

$$H_{gonihedric}^{4d} = - \sum_{\text{pairs of parallel plaquettes}} (\sigma\sigma\sigma\sigma)_P^{\parallel} (\sigma\sigma\sigma\sigma)_P \quad (14)$$

where the summation is extended over all pairs of parallel plaquettes in  $3d$  cubes of the  $4d$  lattice. Here the Ising spins  $\sigma_{\vec{r}, \vec{r}+\vec{\alpha}}$  are located on the center of the links  $(\vec{r}, \vec{r}+\vec{\alpha})$  of the four dimensional lattice. The method of construction of the Hamiltonian (14) from the given distribution of the surface weights is similar to the one which was suggested in [9]. *A posteriori* one can check that the low temperature expansion of the partition function of this system

$$Z(\beta) = \sum_{\{\sigma\}} \exp\{-\beta H_{gonihedric}^{4d}\} \quad (15)$$

is obtained by summation over all closed surfaces  $\{M\}$  with the weight  $\exp(-\beta A(M))$  where the linear action  $A(M)$  is given by the number of nonflat pairs of plaquettes of the closed surface  $M$ .

To construct the dual Hamiltonian let us consider the high temperature expansion of the partition function (15)

$$Z(\beta) = \sum_{\{\sigma\} \text{ pairs of plaquettes}} \prod \{ \cosh \beta + \sinh \beta \cdot (\sigma\sigma\sigma\sigma)_P^{\parallel} (\sigma\sigma\sigma\sigma)_P \} \quad (16)$$

The first nonzero terms of the expansion are (i) a  $3d$  cube covered by the three pairs of plaquettes and (ii) a loop of four pairs of plaquettes forming a torus which can "live" only inside a  $4d$  hypercube. (These terms are those given by eqs. (40) and (43)). To have a natural geometrical representation of all expansion terms, one can associate the link  $l$  connecting the centers of the pairs of parallel plaquettes in a  $3d$  cube. In this language the first term is represented by the "hedgehog" or cross of three perpendicular links inside the  $3d$  cube and we will use  $\Gamma$  for this basic element. The second term is represented by a loop in a  $4d$  hypercube and we will use  $\Lambda$  for this element.

In order to have a complete description of high temperature configurations  $\{\Sigma\}$  let us consider a  $4d$  hypercube. There are eight different configurations  $\Gamma$  and six different configurations  $\Lambda$  inside a  $4d$  hypercube. It is convenient to introduce therefore  $\Gamma_{+\mu}$  and  $\Gamma_{-\mu}$ ,  $\mu = 1, 2, 3, 4$ , for these eight configurations  $\Gamma$  and  $\Lambda_{\mu,\nu} = \Lambda_{\nu,\mu}$ ,  $\mu \neq \nu$  for the six configurations  $\Lambda$ . The indices naturally reflect the positions of the elementary configurations inside the  $4d$  hypercube. Using these basic elements one can construct any configuration  $\Sigma$  of the high temperature expansion (16). The energy of the given configuration  $\Sigma$  is equal to the total length of the links. The configuration  $\Gamma$  has the length equal to three  $((\tanh \beta)^3$  of the high  $T$  expansion) and a  $\Lambda$  has the length equal to four  $((\tanh \beta)^4)$ .

Let us ascribe now spin variables  $\Gamma_{\pm\mu}$  and  $\Lambda_{\mu,\nu}$  to every vertex  $\xi$  of the dual lattice. They will assume the value  $-1$  at the vertex  $\xi$  in the cases when the  $4d$  hypercube is occupied by certain basic element  $\Gamma_{\pm\mu}$  or  $\Lambda_{\mu\nu}$  and  $+1$  if it is not. To every center of the dual lattice we ascribe therefore 14 independent spins  $\Gamma_{\pm\mu}$  and  $\Lambda_{\mu,\nu}(\xi)$ .

There is an ambiguity in the definition of the  $\Gamma_{\pm\mu}$ . One can ascribe configuration  $\Gamma$  to any of two  $4d$  hypercube to which the  $3d$  cube with  $\Gamma$  configuration is the border. This corresponds to a local symmetry which we will clarify later on (see 18).

Now one can construct the dual Hamiltonian

$$H_{dual}^{4d} = \sum_{\xi} \sum_{\mu=1}^4 H(\xi, \xi + e_{\mu}) = - \sum_{\xi} \sum_{\nu \neq \mu} \Lambda_{\mu,\nu}(\xi) \Lambda_{\mu,\nu}(\xi + e_{\mu}) \Gamma_{\mu}(\xi) \Gamma_{-\mu}(\xi + e_{\mu}), \quad (17)$$

where  $e_{\mu}$  is the unite vector in the corresponding direction.

The Hamiltonian (17) has a number of local gauge symmetries: one can independently transform spins  $\Gamma$  in neighbor vertices

$$\Gamma_{\mu}(\xi) \rightarrow -\Gamma_{\mu}(\xi), \quad \Gamma_{-\mu}(\xi + e_{\mu}) \rightarrow -\Gamma_{-\mu}(\xi + e_{\mu}), \quad (18)$$

and flip all spins at a given vertex

$$(\Lambda(\xi), \Gamma(\xi)) \rightarrow -(\Lambda(\xi), \Gamma(\xi)). \quad (19)$$

Using local gauge invariance (18) one can fix the gauge

$$\Gamma_{-\mu}(\xi) = 1 \quad (20)$$

and therefore locate (or fix) the remaining spin  $\Gamma_\mu$  on the *link* between two vertices  $\xi$  and  $\xi + e_\mu$ . It is convenient to use notation  $\Gamma(\xi, \xi + e_\mu)$  for this spin. The fixed gauge Hamiltonian has the form

$$H_{dual}^{4d} = - \sum_{\xi} \sum_{\nu \neq \mu} \Lambda_{\nu, \mu}(\xi) \cdot \Gamma(\xi, \xi + e_\mu) \cdot \Lambda_{\mu, \nu}(\xi + e_\mu) \quad (21)$$

The dual Hamiltonian (21) is therefore a spin system of six Ising spins  $\Lambda_{\mu, \nu}(\xi) = \Lambda_{\nu, \mu}(\xi)$ ,  $\mu \neq \nu = 1, 2, 3, 4$  which are located on every *vertex*  $\xi$  of the lattice and of one Ising spin  $\Gamma(\xi, \xi + e_\mu)$  which is located on the center of every *link*  $(\xi, \xi + e_\mu)$ .

High temperature expansion of the partition function

$$\begin{aligned} Z(\beta^*) &= \sum_{\{\Lambda, \Gamma\}} \exp\{-\beta^* H_{dual}^{4d}\} \\ &= \sum_{\{\Lambda, \Gamma\}} \prod_{\xi} \exp\left\{ \beta^* \sum_{i=2}^4 \Lambda_{1,i}(\xi) \cdot \Gamma(\xi, \xi + e_1) \cdot \Lambda_{1,i}(\xi + e_1) + \dots \right\} \end{aligned} \quad (22)$$

generate random surfaces with linear action  $A(M)$ . Indeed, expanding (22) for small  $\beta^*$

$$Z(\beta^*) = \sum_{\{\Lambda, \Gamma\}} \prod_{\xi} \prod_{i=2}^4 (\cosh \beta^*)^4 \{1 + \tanh \beta^* \cdot \Lambda_{1,i}(\xi) \cdot \Gamma(\xi, \xi + e_1) \cdot \Lambda_{1,i}(\xi + e_1)\} \dots \quad (23)$$

and then summing over  $\Gamma(\xi, \xi + e_1)$  and computing the product over  $i$  yields

$$\begin{aligned} Z(\beta^*) &= \sum_{\{\Lambda\}} \prod_{\xi} (\cosh \beta^*)^4 2 \cdot \{ 1 + (\tanh \beta^*)^2 \cdot \\ &\quad [\Lambda_{1,2}(\xi) \Lambda_{1,3}(\xi) \Lambda_{1,2}(\xi + e_1) \Lambda_{1,3}(\xi + e_1) + \\ &\quad \Lambda_{1,3}(\xi) \Lambda_{1,4}(\xi) \Lambda_{1,3}(\xi + e_1) \Lambda_{1,4}(\xi + e_1) + \\ &\quad \Lambda_{1,4}(\xi) \Lambda_{1,2}(\xi) \Lambda_{1,4}(\xi + e_1) \Lambda_{1,2}(\xi + e_1)] \} \dots \end{aligned} \quad (24)$$

Opening the brackets and summing over spins  $\Lambda$  one can see that nonzero terms are those which correspond to closed surfaces  $M$  with linear action  $A(M)$ .

Both Hamiltonian (5) and (17) look differently, but it is possible to rederive (5) in the form which is similar to (17), (21). For that let us introduce three different Ising spins  $\{\Lambda_1, \Lambda_2, \Lambda_3\}$  in every vertex  $\xi$ , then

$$H_{dual}^{3d} = \sum_{\xi} \sum_{i=1}^3 H(\xi, \xi + e_i) = \sum_{i \neq j \neq k} \Lambda_j(\xi) \Lambda_k(\xi) \Lambda_j(\xi + e_i) \Lambda_k(\xi + e_i) \quad (25)$$

and we observe the local gauge symmetry of the Hamiltonian (25)

$$\Lambda_{1,2,3}(\xi) \rightarrow -\Lambda_{1,2,3}(\xi) \quad (26)$$

### 3 Surfaces with linear action in high dimensions

#### 3.1 The Hamiltonian

In [9] the authors have introduced also a model of  $(d-n)$ -dimensional hypersurfaces on a  $d$ -dimensional hypercubic lattice. Such a hypersurface  $M_{d-n}$  consists of a collection of elementary hyperplaquettes  $\Omega_{\alpha_1 \dots \alpha_n}(\vec{r})$  (all  $\alpha_i$  are different) with  $\vec{r}$  on a simple cubic lattice  $Z^d$ . These are defined by  $x_{\alpha_i} = r_{\alpha_i}$  and  $r_\alpha \leq x_\alpha \leq r_\alpha + 1$  for all other  $\alpha$ . As in [9] we introduce a variable  $U_{\alpha_1 \dots \alpha_n}(\vec{r})$ , which assumes the value  $-1$  if the hyperplaquette belongs to the hypersurface  $M_{d-n}$  and  $+1$  if it does not.

The hypersurface  $M_{d-n}$  shall be closed. That is each  $(d-n-1)$ -dimensional hyperedge  $\Omega_{\alpha_1 \dots \alpha_{n+1}}(\vec{r})$  belongs totally to an even number of  $(d-n)$ -dimensional hyperplaquettes of the hypersurface  $M_{d-n}$ . This imposes the constraint

$$\prod_{k=1}^{n+1} U_{\alpha_1 \dots \alpha_{k-1} \alpha_{k+1} \dots \alpha_{n+1}}(\vec{r}) U_{\alpha_1 \dots \alpha_{k-1} \alpha_{k+1} \dots \alpha_{n+1}}(\vec{r} - \vec{e}_{\alpha_k}) = 1 \quad (27)$$

We associate to each hypersurface  $M$  a weight  $\exp(-H(M))/Z$ , where  $Z = \sum_M \exp(-H(M))$  is the partition function of the system. We suppose that the energy  $H$  can be written as a sum of terms depending on the configuration around each hyperedge, more precisely

$$H_{gonihedric}^d = \sum_{\alpha_1 < \dots < \alpha_{n+1}, \vec{r}} H_{\alpha_1 \dots \alpha_{n+1}}(\vec{r}), \quad (28)$$

where  $H_{\alpha_1 \dots \alpha_{n+1}}(\vec{r})$  depends on the  $U$ 's of the hyperplaquettes by which the hyperedge is surrounded. These are the  $U$ 's in eq. (27).

In [9] the Hamiltonian counted the number of right angles at all hyperedges (eq.(20) of [9]). Here we will count the number of hyperplaquettes  $\Omega_{\alpha_1 \dots \alpha_n}(\vec{r})$  in  $M_{d-n}$  which do not have a straight continuation at a hyperedge. If we multiply this number by  $2K$ , then the contribution of a hyperedge reads

$$H_{\alpha_1 \dots \alpha_{n+1}}(\vec{r}) = K \sum_k (1 - U_{\alpha_1 \dots \alpha_{k-1} \alpha_{k+1} \dots \alpha_{n+1}}(\vec{r}) U_{\alpha_1 \dots \alpha_{k-1} \alpha_{k+1} \dots \alpha_{n+1}}(\vec{r} - \vec{e}_{\alpha_k})). \quad (29)$$

In the following we will skip the unimportant constant. Then we may write

$$H_{\alpha_1 \dots \alpha_{n+1}}(\vec{r}) = -K \sum_k V_{\alpha_1 \dots \alpha_{k-1} \alpha_{k+1} \dots \alpha_{n+1}, \alpha_k}(\vec{r}). \quad (30)$$

with

$$V_{\alpha_1 \dots \alpha_{k-1} \alpha_{k+1} \dots \alpha_{n+1}, \alpha_k}(\vec{r}) = U_{\alpha_1 \dots \alpha_{k-1} \alpha_{k+1} \dots \alpha_{n+1}}(\vec{r}) U_{\alpha_1 \dots \alpha_{k-1} \alpha_{k+1} \dots \alpha_{n+1}}(\vec{r} - \vec{e}_{\alpha_k}). \quad (31)$$

In this formulation the energy associated with a simple kink at a hyperedge is  $4K$  times the length of the kink. As usual [13] the constraint (27) can be eliminated by the introduction of Ising spins  $\sigma$  attached to the  $(d-n+1)$  dimensional hypercubes  $\Omega_{\alpha_1 \dots \alpha_{n-1}}$ . Then we obtain for  $V$  the product over the spins at the boundary of two neighbouring  $(d-n)$ -dimensional hyperplaquettes which are parallel to each other

$$\begin{aligned} V_{\alpha_1 \dots \alpha_n, \beta}(\vec{r}) &= \prod_{k=1}^n \sigma_{\alpha_1 \dots \alpha_{k-1} \alpha_{k+1} \dots \alpha_n}(\vec{r}) \sigma_{\alpha_1 \dots \alpha_{k-1} \alpha_{k+1} \dots \alpha_n}(\vec{r} - \vec{e}_{\alpha_k}) \\ &\quad \sigma_{\alpha_1 \dots \alpha_{k-1} \alpha_{k+1} \dots \alpha_n}(\vec{r} - \vec{e}_\beta) \sigma_{\alpha_1 \dots \alpha_{k-1} \alpha_{k+1} \dots \alpha_n}(\vec{r} - \vec{e}_{\alpha_k} - \vec{e}_\beta). \end{aligned}$$



### 3.2 Duality transformation

First we describe the general algebraic procedure along the lines of [18] which allows to obtain the dual model of an Ising model. For this purpose the Hamiltonian is written as a linear combination of Ising variables as in (30)

$$H = - \sum_i K_i V_i, \quad (32)$$

where for the moment we denote the products of the Ising variables by  $V_i$ . They cannot assume values  $\pm 1$  independently from each other, for example due to the constraint (27). One has to find a complete set of constraints for the  $V$ 's so that any set of spins  $V$  which obeys these constraints yields at least one allowed configuration of  $U$ 's. These constraints have to have the form

$$\prod_i V_i^{m_{ij}} = 1, \quad m_{i,j} = 0, 1. \quad (33)$$

Then the partition function can be written

$$Z = 2^n \sum_{\{\sigma\}, \{V\}} \prod_i V_i^{\sum_j m_{ij} \sigma_j} e^{K_i V_i}. \quad (34)$$

With each constraint one variable  $\sigma_j = 0, 1$  has been introduced. If the constraint (33) is violated, that is the left hand-side of this equation equals  $-1$  then the sum over  $\sigma_j$  yields zero, but if this fulfilled it yields twice the contribution. This factor of two is taken care of in the prefactor  $2^n$ . Thus the summation over the  $V$ 's can be performed independently without any constraints.

For a given configuration of the  $V$ 's there are in general  $2^{n_d}$  configuration of the  $U$ 's. Normally this is the degeneracy of the groundstate. The exponent  $n$  is the number  $n_d$  minus the number of constraints.

With the  $\sigma$ 's we associate the Ising variables

$$S_j = (-)^{\sigma_j} \quad (35)$$

of the dual model. Now the sum over each  $V_i = \pm 1$  can be evaluated,

$$e^{K_i} + e^{-K_i} \prod_j S_j^{m_{ij}} = \sinh(2K_i) \exp(-K_i^* \prod_j S_j^{m_{ij}}) \quad (36)$$

with

$$\exp(-2K_i^*) = \tanh(K_i). \quad (37)$$

Thus the partition function  $Z$  is expressed

$$Z = 2^n \prod_i \sinh(2K_i) Z^* \quad (38)$$

by the partition function  $Z^*$  of the dual model

$$H^* = - \sum_i K_i^* \prod_j S_j^{m_{ij}}. \quad (39)$$

It may be, that not all of the constraints (33) are chosen independently. More precisely it may be, that they are multiplicatively dependent, that is there exist numbers  $n_j$ , at least one of which is odd, so that the sum  $\sum_j n_j m_{ij}$  is an even number for all  $i$ . Then the product of the conditions (33) with odd  $j$  is identically 1, since each factor  $V$  appears an even number of times. In this case we obtain more Ising spins  $S$  than necessary. Then the dual Hamiltonian is invariant under the transformation  $S_j \rightarrow (-)^{n_j} S_j$ . If there are sets of  $n_j$  which differ from zero only in a small region, then this is a gauge transformation.

### 3.3 Application to hypersurfaces with linear action

An obvious constraint is given by eq. (27). It can be written in terms of the  $V$ 's as

$$\prod_{k=1}^{n+1} V_{\alpha_1 \dots \alpha_{k-1} \alpha_{k+1} \dots \alpha_{n+1}, \alpha_k}(\vec{r}) = 1. \quad (40)$$

To find further constraints let us assume that we know the  $U(\vec{r})$  for  $\sum_{\alpha=1}^d r_\alpha < c$  which are compatible with the  $V$  and now we calculate the  $U(\vec{r})$  with  $\sum_{\alpha=1}^d r_\alpha = c$  from those and the  $V$ 's. Then apparently from the definition of  $V$ , eq. (31) we obtain

$$U_{\alpha_1 \dots \alpha_n}(\vec{r}) = V_{\alpha_1 \dots \alpha_n, \beta}(\vec{r}) U_{\alpha_1 \dots \alpha_n}(\vec{r} - \vec{e}_\beta). \quad (41)$$

For  $n = d - 1$  there is only one choice of  $\beta$ , since the  $\alpha$  and  $\beta$  have all to be different. For  $n < d - 1$  however we have different choices for  $\beta$ . Let us choose  $\gamma$  instead. Then a solution requires

$$V_{\alpha_1 \dots \alpha_n, \beta}(\vec{r}) U_{\alpha_1 \dots \alpha_n}(\vec{r} - \vec{e}_\beta) = V_{\alpha_1 \dots \alpha_n, \gamma}(\vec{r}) U_{\alpha_1 \dots \alpha_n}(\vec{r} - \vec{e}_\gamma). \quad (42)$$

This is equivalent to the condition

$$V_{\alpha_1 \dots \alpha_n, \beta}(\vec{r}) V_{\alpha_1 \dots \alpha_n, \gamma}(\vec{r}) V_{\alpha_1 \dots \alpha_n, \beta}(\vec{r} - \vec{e}_\gamma) V_{\alpha_1 \dots \alpha_n, \gamma}(\vec{r} - \vec{e}_\beta) = 1 \quad (43)$$

We neglect any constraints due to the boundary condition. We expect, that they do not contribute in the thermodynamic limit.

According to (35) we introduce for each constraint an Ising spin on the dual lattice. We denote the Ising spin for the condition (40)  $\Gamma_{\alpha_1 \dots \alpha_{n+1}}(\vec{r})$ . Apparently it is invariant under the permutations of all  $\alpha$ 's, since the condition (40) has this property. For the constraint (43) we introduce the Ising spins  $\Lambda_{\alpha_1 \dots \alpha_n, \beta \gamma}(\vec{r})$ , which is invariant under the permutation of all  $\alpha$ 's and under the exchange of  $\beta$  and  $\gamma$ . Now to each of the original terms  $K V_{\alpha_1 \dots \alpha_{k-1} \alpha_{k+1} \dots \alpha_{n+1}, \alpha_k}(\vec{r})$  there corresponds a term in the dual Hamiltonian  $H^* \equiv K H_{dual}^d$ . To obtain this term we have to find all constraints in which  $V_{\alpha_1 \dots \alpha_{k-1} \alpha_{k+1} \dots \alpha_{n+1}, \alpha_k}(\vec{r})$  appears and to multiply the corresponding Ising spins on the dual lattice. This yields the dual Hamiltonian

$$H_{dual}^d = - \sum_{\vec{r}} \sum_{\alpha_1 < \dots < \alpha_n, \beta \neq \alpha_i} \Gamma_{\alpha_1 \dots \alpha_n, \beta}(\vec{r}) \prod_{\gamma \neq \alpha_i, \beta} \Lambda_{\alpha_1 \dots \alpha_n, \beta \gamma}(\vec{r}) \Lambda_{\alpha_1 \dots \alpha_n, \beta \gamma}(\vec{r} + \vec{e}_\gamma). \quad (44)$$

### 3.4 Geometrical interpretation

Thus we have to multiply  $d - n - 1$  factors  $\Lambda(\vec{r})$  with the corresponding factors  $\Lambda$  at sites  $\vec{r} + \vec{e}_\gamma$  and this interaction is mediated by an Ising spin  $\Gamma(\vec{r})$ . The spins  $\Gamma_{\alpha_1 \dots \alpha_n \beta}(\vec{r})$  can be considered to be located on the hyperedges  $\Omega_{\alpha_1 \dots \alpha_n \beta}(\vec{r})$ . They are bounded by  $(d - n - 2)$  dimensional hypervertex  $\Omega_{\alpha_1 \dots \alpha_n \beta \gamma}(\vec{r})$  and  $\Omega_{\alpha_1 \dots \alpha_n \beta \gamma}(\vec{r} + \vec{e}_\gamma)$  at which one may consider the location of the Ising spins  $\Lambda_{\alpha_1 \dots \alpha_n, \beta \gamma}(\vec{r})$  which are multiplied by  $\Gamma$ . Note however, that at each of these hypervertex there is the location of  $(n + 2)(n + 1)/2$  different spins  $\Lambda$ .

Let us give a geometrical interpretation of the duality transformation by comparing the high-temperature expansion of the dual model (44) with the low-temperature expansion of the original model (28), (30), (31). In the high-temperature expansion one writes each factor

$$\exp(K^* \prod S) = \cosh(K^*) (1 + \tanh(K^*) \prod S) \quad (45)$$

and expands in powers of  $\tanh(K^*)$ . Only those products over  $S$  contribute to the sum over  $S$  in which each  $S$  appears an even number of times. Let us consider the products of  $\Lambda_{\alpha_1 \dots \alpha_n, \beta \gamma}$  from the dual Hamiltonian (44) for fixed  $\alpha_1 \dots \alpha_n$ . Each term consists of  $2(d - n - 1)$  factors  $\Lambda$  which are located at the boundaries of a  $(d - n - 1)$ -dimensional hypercube. In order that each factor  $\Lambda$  appears pairwise each boundary of these hypercubes has to appear for an even number of times. Thus it constitutes a closed  $(d - n - 1)$ -dimensional closed hypersurface, which encloses a flat  $(d - n)$ -dimensional hypersurface. At each of these  $(d - n - 1)$ -dimensional hypercubes there is a factor  $\Gamma$ . They again have to appear pairwise. Thus these  $(d - n - 1)$ -dimensional hypercubes are the edges of a closed  $(d - n)$ -dimensional hypersurface. This hypersurface contributes to the partition function with a term  $\tanh(K^*)$  to the power twice the number of elementary edge pieces. Thus the partition function agrees with that of our original model provided

$$(\tanh(K^*))^2 = \exp(-4K) \quad (46)$$

in agreement with eq.(37). For fixed  $\Gamma$  the model decays into models for spins  $\Lambda$  with fixed  $\alpha_1 \dots \alpha_n$  in  $n$ -dimensional hyperplanes with fixed coordinates  $x_{\alpha_1}$  to  $x_{\alpha_n}$ . The spins  $\Gamma$  provide a coupling between these models.

### 3.5 Gauge transformations

In general the model will be gauge invariant. Let us determine the number of gauge degrees of freedom per  $d$ -dimensional elementary cell (hypercube). The number  $n_{iU}$  of independent variables  $U$  per elementary cell is

$$n_{iU} = \binom{d-1}{n-1} \quad (47)$$

since for given  $U(\vec{r} - \vec{e})$  and  $U_{1\alpha_2 \dots \alpha_n}(\vec{r})$  the other  $U_{\alpha_1 \dots \alpha_n}(\vec{r})$  are obtained uniquely from the constraint (27). Thus  $n_{iU}$  out of the

$$n_V = (d - n) \binom{d}{n} \quad (48)$$

variables  $V$  are independent. For the duality transformation  $n_\Gamma$  and  $n_\Lambda$  constraints are used, where  $n_\Gamma$  and  $n_\Lambda$  are the numbers of spins  $\Gamma$  and  $\Lambda$  per elementary cell,

$$n_\Gamma = \binom{d}{n+1}, \quad n_\Lambda = \frac{(d-n)(d-n-1)}{2} \binom{d}{n}. \quad (49)$$

Thus

$$n_g = n_\Gamma + n_\Lambda - n_V + n_{iU} \quad (50)$$

is the number of gauge degrees of freedom per elementary cell.

A first class of gauge transformations can be easily given for  $n \leq d-3$ . The simultaneous switching of the Ising spins  $\Lambda_{\alpha_1 \dots \alpha_n, \beta \gamma}(\vec{r})$ ,  $\Lambda_{\alpha_1 \dots \alpha_n, \beta \delta}(\vec{r})$ ,  $\Lambda_{\alpha_1 \dots \alpha_n, \gamma \delta}(\vec{r})$ ,  $\Lambda_{\alpha_1 \dots \alpha_n, \beta \gamma}(\vec{r} - \vec{e}_\delta)$ ,  $\Lambda_{\alpha_1 \dots \alpha_n, \beta \delta}(\vec{r} - \vec{e}_\gamma)$ ,  $\Lambda_{\alpha_1 \dots \alpha_n, \gamma \delta}(\vec{r} - \vec{e}_\beta)$  leaves the dual Hamiltonian invariant. We denote this gauge transformation by  $G_{\alpha_1 \dots \alpha_n, \beta \gamma \delta}^{(1)}(\vec{r})$ . It is invariant under permutation of the  $\alpha$ 's and separately of  $\beta$ ,  $\gamma$  and  $\delta$ . These gauge transformations are not independent from each other for  $n \leq d-4$ , since the application of the following eight transformations leave all spins invariant,

$$G_{\alpha_1 \dots \alpha_n, \beta \gamma \delta}^{(1)}(\vec{r}) G_{\alpha_1 \dots \alpha_n, \beta \gamma \epsilon}^{(1)}(\vec{r}) G_{\alpha_1 \dots \alpha_n, \beta \delta \epsilon}^{(1)}(\vec{r}) G_{\alpha_1 \dots \alpha_n, \gamma \delta \epsilon}^{(1)}(\vec{r}) \\ G_{\alpha_1 \dots \alpha_n, \beta \gamma \delta}^{(1)}(\vec{r} - \vec{e}_\epsilon) G_{\alpha_1 \dots \alpha_n, \beta \gamma \epsilon}^{(1)}(\vec{r} - \vec{e}_\delta) G_{\alpha_1 \dots \alpha_n, \beta \delta \epsilon}^{(1)}(\vec{r} - \vec{e}_\gamma) G_{\alpha_1 \dots \alpha_n, \gamma \delta \epsilon}^{(1)}(\vec{r} - \vec{e}_\beta) = 1.$$

These gauge transformations are independent, if one chooses for given  $\alpha$ 's the smallest possible  $\beta$ . Those with larger  $\beta$ 's can be constructed from the first ones by use of eq. (51). Thus one has

$$n_{g1} = \frac{(d-n-1)(d-n-2)}{2} \binom{d}{n} \quad (51)$$

independent gauge degrees of freedom from  $G^{(1)}$ .

A second class of gauge transformations denoted by  $G_{\alpha_1 \dots \alpha_{n+2}}^{(2)}(\vec{r})$  exists for  $n \leq d-2$  and involves also the  $\Gamma$ 's. It is invariant under all permutations of the  $\alpha$ 's. This transformation switches the spins  $\Lambda_{\alpha_1 \dots \alpha_{k-1} \alpha_{k+1} \dots \alpha_{l-1} \alpha_{l+1} \dots \alpha_{n+2}, \alpha_k \alpha_l}(\vec{r})$  for all  $k < l$ ,  $\Gamma_{\alpha_1 \dots \alpha_{k-1} \alpha_{k+1} \dots \alpha_{n+1}}(\vec{r} - \vec{e}_{\alpha_k})$  and  $\Gamma_{\alpha_1 \dots \alpha_{k-1} \alpha_{k+1} \dots \alpha_{n+1}}(\vec{r})$  for all  $k$ . Again these gauge transformations are not independent for  $n \leq d-3$ , since

$$\prod_{k=1}^{n+3} G_{\alpha_1 \dots \alpha_{k-1} \alpha_{k+1} \dots \alpha_{n+3}}^{(2)}(\vec{r}) G_{\alpha_1 \dots \alpha_{k-1} \alpha_{k+1} \dots \alpha_{n+3}}^{(2)}(\vec{r} - \vec{e}_{\alpha_k}) \\ \prod_{k < l < m} G_{\alpha_1 \dots \alpha_{k-1} \alpha_{k+1} \dots \alpha_{l-1} \alpha_{l+1} \dots \alpha_{m-1} \alpha_{m+1} \dots \alpha_{n+3}, \alpha_k \alpha_l \alpha_m}^{(1)}(\vec{r}) = 1.$$

By means of this equation we can express all  $G_{\alpha_1 \dots \alpha_{n+2}}^{(2)}(\vec{r})$  with  $\alpha_i > 1$  for all  $i$  by those where one  $\alpha$  equals 1 and  $G^{(2)}(\vec{r} - \vec{e})$  and by  $G^{(1)}$ 's. The number of  $G^{(2)}$ 's with  $\alpha_1 = 1$  per lattice site is

$$n_{g2} = \binom{d-1}{n+1}. \quad (52)$$

They are independent gauge transformations. Since

$$n_g = n_{g1} + n_{g2}. \quad (53)$$

these two types of gauge transformations exhaust all independent gauge transformations.

Now we will consider some special cases.

### 3.6 Two-dimensional surfaces

In the case  $d - n = 2$  we have two-dimensional surfaces  $M_2$ . If we denote the  $\Gamma_{\alpha_1 \dots \alpha_{d-2} \beta}(\vec{r})$  by  $\Gamma(\vec{r}, \vec{r} + \vec{e}_\gamma)$  where  $\gamma$  is the index, that is different from the  $\alpha_i$  and  $\beta$ , and the  $\Lambda_{\alpha_1 \dots \alpha_{d-2}, \beta \gamma}$  by  $\Lambda_{\beta \gamma}$ , then the Hamiltonian reads

$$H_{dual}^d = - \sum_{\vec{r}} \sum_{\beta \neq \gamma} \Lambda_{\beta \gamma}(\vec{r}) \Gamma(\vec{r}, \vec{r} + \vec{e}_\gamma) \Lambda_{\beta \gamma}(\vec{r} + \vec{e}_\gamma). \quad (54)$$

For fixed  $\Gamma$  this model can be regarded as a model of two-dimensional Ising-models in plains parallel to the  $\beta, \gamma$ -plain. The  $\Gamma$ 's produce a coupling between these Ising-models. Here we have one gauge degree of freedom  $G_{1 \dots d}^{(2)}(\vec{r})$  per elementary cell. Indeed the dual Hamiltonian is invariant under simultaneous switching of all  $\Lambda_{\beta \gamma}(\vec{r})$ ,  $\Gamma(\vec{r} - \vec{e}_\gamma, \vec{r})$  and  $\Gamma(\vec{r}, \vec{r} + \vec{e}_\gamma)$  for fixed  $\vec{r}$ .

### 3.7 Hypersurfaces of codimension one

In this case of a  $d - 1$  dimensional hypersurface  $M_{d-1}$  the constraint (40) reads

$$V_{\alpha, \beta}(\vec{r}) V_{\beta, \alpha}(\vec{r}) = 1, \quad (55)$$

thus  $V_{\alpha, \beta}(\vec{r}) = V_{\beta, \alpha}(\vec{r})$ . If one considers these two  $V$ 's to be identical, then the constraint (40) and the Ising spin  $\Gamma_{\alpha \beta}$  disappears. Then the original Hamiltonian reads

$$H = -2K \sum_{\alpha < \beta, r} V_{\alpha, \beta}(\vec{r}) \quad (56)$$

and  $V_{\alpha, \beta}(\vec{r})$  does not only appear in the constraint associated with  $\Lambda_{\alpha, \beta \gamma}(\vec{r})$  and  $\Lambda_{\alpha, \beta \gamma}(\vec{r} + \vec{e}_\gamma)$  but also in the conditions for  $\Lambda_{\beta, \alpha \gamma}(\vec{r})$  and  $\Lambda_{\beta, \alpha \gamma}(\vec{r} + \vec{e}_\gamma)$ . This yields

$$H_2^* = -K_2^* \sum_{\alpha < \beta, r} \prod_{\gamma} \Lambda_{\alpha, \beta \gamma}(\vec{r}) \Lambda_{\alpha, \beta \gamma}(\vec{r} + \vec{e}_\gamma) \Lambda_{\beta, \alpha \gamma}(\vec{r}) \Lambda_{\beta, \alpha \gamma}(\vec{r} + \vec{e}_\gamma) \quad (57)$$

with

$$\exp(-2K_2^*) = \tanh(2K). \quad (58)$$

This expression can also be obtained from  $H^*$  by summing over the  $\Gamma$

$$\exp(-H_2^*) = \text{const.} \sum_{\Gamma} \exp(-H^*) \quad (59)$$

where use is made, that a given  $\Gamma$  appears only in two terms  $\Gamma \Lambda^{2(d-n-1)}$  in  $H^*$ . This elimination of  $\Gamma$  is a modification of the decoration transformation [19].

### 3.8 Hypersurfaces of codimension two

In this case of a  $d - 2$  dimensional hypersurface  $M_{d-2}$  a given  $\Gamma_{\alpha_1 \alpha_2 \alpha_3}(\vec{r})$  is multiplied by the three terms

$$A_i = \prod_{\gamma} (\Lambda_{\alpha_i \alpha_{i+1}, \alpha_{i+2} \gamma}(\vec{r}) \Lambda_{\alpha_i \alpha_{i+1}, \alpha_{i+2} \gamma}(\vec{r} + \vec{e}_\gamma)). \quad (60)$$

Here the indices  $i + 1$  and  $i + 2$  are meant modulo 3. Then summation over the  $\Gamma$ 's which is analogous to the star-triangle transformation [20] yields

$$-H_3^* = K_3^* \sum (A_1 A_2 + A_1 A_3 + A_2 A_3). \quad (61)$$

with

$$\exp(2K_3^*) = \exp(2K^*) - 1 + \exp(-2K^*). \quad (62)$$

For larger values of  $n$  one can also eliminate  $\Gamma$ . Then however the new interaction involves not only bilinear terms in the  $A$ 's, but also terms of higher order in the  $A$ 's.

### 3.9 Random walks

In the case  $d - n = 1$  we have the model of closed random walks  $M_1$ . There are no variables  $\Lambda$ , since there is no constraint (43). Then there is only one Ising spin  $\Gamma$  at each lattice site appearing in  $d$  terms. The dual Hamiltonian simply reads

$$H^* = -K^* d \sum_{\vec{r}} \Gamma(\vec{r}). \quad (63)$$

Thus there is no interaction between the spins. They are only subject to an external field  $K^* d$ . Obviously there is no phase transition in this case. The partition function reads

$$Z^* = (2 \cosh(dK^*))^N \quad (64)$$

where  $N$  is the number of elementary cells. In this model there is no gauge invariance.

## Conclusion and Acknowledgements

In this paper we describe the lattice implementation of the string with linear action which allows self-intersections. For this model not only an equivalent Ising-spin model is given, but also a dual model was constructed. The generalization for  $p$ -membranes is given. In general the models have gauge degrees of freedom.

We should mention that the dual representations are important in our attempt to understand confinement phase of the gauge theory [23, 24, 25, 26]. Recent results in this direction can be found in [27, 28].

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*Note added*

We are thankful to the referee for pointing out an important article of A.Cappi, P.Colangelo, G.Gonnella and A.Maritan in Nucl.Phys. B370 (1992) 659, where the authors consider the general three-dimensional spin model with straight, diagonal and square spin interaction. In terms of interface surfaces the model contains an area term  $\beta_s$ , the gonihedric term  $\beta_c$  and self-intersection term  $\beta_l$ .

The special case, which we discuss in three dimensions, corresponds to  $\beta_s = 0$ ,  $\beta_c = 1$  and  $\beta_l = 4k - 2$ , with  $k = 0$ . The principle point is that  $\beta_s = 0$  and the case  $k = 0$  corresponds to the "super symmetric" point (0,1,-2) in the space of Hamiltonians  $(\beta_s, \beta_c, \beta_l)$ , because at that point the degeneracy of the vacuum is extremely high and is equal to  $2^{dN}$  for the lattice of the size  $N^d$ . The curvature representation shows that the systems with and without area term are in different class of universality (see [12] and section 1.4). String interpretation of this spin systems requires the existence of the second order phase transition and an appropriate continuum limit at the critical point.

Second remark concerns the non-Abelian extension of the two-plaquette action (14). If we exchange the spin variable  $\sigma$  by the element of the  $SU(N)$ ,  $U = \exp(igaA)$ , then the Hamiltonian takes the form

$$H = - \sum_{\text{pairs of parallel plaquettes}} \frac{\beta}{n^2} (Tr U_{\square} + Tr U_{\square}^*) (Tr U_{\square} + Tr U_{\square}^*) \quad (65)$$

and in the naive continuum limit reproduces the usual Yang-Mills action.

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